DISQUISITIO ACADEMICA,

INTEGRATIONEM ÆQUATIONIS CUJUSDAM DIFFERENTIALIS EXHIBENS,

QUAM

CONSENT. AMPL. ORD. PHILOS. LUNDENS.,

PRÆSIDE

CAROLO JOH. Ds HILL,

MATH. PROF. REG. ET ORD.,

PRO GRADU PHILOSOPHICO

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CAROLUS ALB. EHRENSVÄRD,

COMES.

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LUNDÆ,
TYPIS EXCUDIT CAROLUS FR. BERLING,
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PATRI OPTIMO, CARISSIMO

IN DIE SUO NATALI LV:mo

SACRUM VOLUIT, DEBUIT

PIUS FILIUS.

Problema I.

Æquationem differentialem (2:di-gradus)
$A.(y'dx'-x'dy')+Bdx'+Cdy'=o$, ubi $A=a_0+a_{,}x'+a_{,,}y'$, $B=b_0+b_{,}x'+b_{,,}y'$ & $C=c_0+c_{,}x'+c_{,,}y'$, & coefficientes ipsorum x' & y' constantes sunt, *integrare.*

Primum patet, numerum a_0 tolli, scribendo $a_0+b_{,,}$ loco ipsius $b_{,,}$ & $c_{,}-a_0$ pro $c_{,}$. Sit igitur jam $a_0=o$. Ponatur deinde $x'=x+x_0$ & $y'=y+y_0$, eritque nostra æquatio

$$\left.\begin{array}{r}(a_{,}x+a_{,,}y)(ydx-xdy)+(a_{,}x_0+a_{,,}y_0)(ydx-xdy)\\ +(a_{,}x+a_{,,}y)(y_0dx-x_0dy)\\ +(a_{,}x_0+a_{,,}y_0)(y_0dx-x_0dy)\\ +B'dx+C'dy+B^\circ dx+C^\circ dy\end{array}\right\}=o$$

si $B'=b_{,}x+b_{,,}y$, $B^\circ=b_0+b_{,}x_0+b_{,,}y_0$, & $C'=c_{,}x+c_{,,}y$, $C^\circ=c_0+c_{,}x_0+c_{,,}y_0$. Jam vero x_0 & y_0 ita algebraice et quidem cubice accipi possunt, ut $B^\circ+(a_{,}x_0+a_{,,}y_0)y_0=o$ & $C^\circ-(a_{,}x_0+a_{,,}y_0)x_0=o$ sint; quo facto nostra æquatio erit *dihomogenea* *) & ejusdem formæ, ac si jam ab initio & b_0 & c_0

*) I. e. formæ $(1^c xy+2^c xy)dy+(I^c xy+II^c xy)dx=o$, si universim $n^c xy$ functionem homogeneam n:ti gradus indicat, et igitur $I^c xy$ et $1^c xy$ primi atque $2^c xy$ et $II^c xy$ secundi gradus sunt functiones homogeneæ.

defuissent. Hoc vero universim tribus diversis efficitur modis. Posito enim $y_0 = r.x_0$, invenitur $x_0 = -\dfrac{b_0 + rc_0}{b_{,} + (b_{,,} + c_{,})r + c_{,,}r^2}$ atque r ex æquatione cubica: **).

$(b_{,} + \overline{b_{,,} + c_{,}}\,r + c_{,,}r^2)(\overline{c_0 b_{,}} + \overline{c_0 b_{,,}}\,r) = (a_{,} + a_{,,}r)\cdot(b_0 + rc_0)^2$, cujus una saltem radix r, ideoque unum systema valorum x_0 & y_0, realis est: nisi forte hæc æquatio in quadraticam degeneraret (ex ratione numerorum $c_{,,}\overline{c_0 b_{,,}} = a_{,,}c_0^2$), quo tamen casu formalis saltem solutio daretur.

Sint igitur jam $b_0 = o = c_0$, & ponantur $y = \dfrac{1}{u}$ & $x = \dfrac{z'}{u}$, quo facto æquatio proposita $(a_{,}x + a_{,,}y)(ydx - xdy) + B'dx + C'dy = o$ migrabit in hanc integrabilem $\dfrac{a_{,}z' + a_{,,}}{u}\cdot\dfrac{dz'}{u^2} + \dfrac{b_{,}z' + b_{,,}}{u}\cdot\dfrac{udz' - z'du}{u^2} - \dfrac{c_{,}z' + c_{,,}}{u}\cdot\dfrac{du}{u^2} = o$ seu $(a_{,}z' + a_{,,} + (b_{,}z' + b_{,,})u)dz' = (b_{,}z'^2 + b_{,,}z' + c_{,}z' + c_{,,})du$. Jam vero, nisi $b_{,} = o$ fuerit, hæc facile (ponendo $b_{,}z' = z + z_0$) in hanc mutatur formam $(æz + a + (z + b)u)\,dz = (z^2 - e^2)du$, seu $du - \dfrac{(z+b)dz}{z^2 - e^2}.u = \dfrac{æz + a}{z^2 - e^2}dz$. †)

Quia vero $\displaystyle\int\frac{(z+b)dz}{z^2 - e^2} = LZ$, si $Z = \overline{z - e}^{\frac{1+n}{2}}\cdot\overline{z + e}^{\frac{1-n}{2}}$ & $b = ne$, & factor ido-

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**) Universim per $\overline{c_0 b_{,}}$ formulam $c_0 b_{,} - c_{,}b_0$, atque $a_0\overline{b_{,}c_{,,}} - b_0\overline{a_{,}c_{,,}} + c_0\overline{a_{,}b_{,,}}$ per $\overline{a_0 b_{,} c_{,,}}$, quæ linearium æquationum sunt *determinantes* et interdum *permutationum negativarum* nomine celebrantur, indicamus.

†) Restituendo enim $b_{,}z' - z_0$ pro z, invenitur $(æb_{,}z' - æz_0 + a + (b_{,}z' - z_0 + b)u)b_{,}dz' = (b_{,}^2 z'^2 - 2b_{,}z_0 z' + z_0^2 - e^2)du$, ideoque hanc comparando æquationem cum illa antea inventa atque in $b_{,}$ ducta, erit $a_{,} = æb_{,}$, $a_{,,} = a - æz_0$, $b_{,,} = b - z_0$, $c_{,} + b_{,,} = -2z_0$, et $b_{,}c_{,,} = z_0^2 - e^2$, unde facile petas $æ = \dfrac{a_{,}}{b_{,}}$, $a = a_{,,} + \dfrac{a_{,}}{b_{,}}.z_0$, $b = b_{,,} + z_0$, $z_0 = -\dfrac{b_{,,} + c_{,}}{2}$, atque $e = \sqrt{z_0^2 - c_{,,}b_{,}}$.

nens ipsius $du-\frac{udZ}{Z}$ est $\frac{1}{Z}$, erit ducendo in $\frac{1}{Z}$, & integrando $\frac{u}{Z}=\int\frac{(\alpha z+a).dz}{(z^2-e^2)Z}$ $=\int\frac{(\alpha z+a)dz}{(z-e)^{\frac{3+n}{2}}.\overline{z+e}^{\frac{3-n}{2}}}+Cst.$ Invento vero ita u in z', habebuntur & y (utut $=\frac{1}{u}$) & $x(=\frac{z'}{u})$ in z'; sicuti inverse const. in x et y, (nempe positis $\frac{x}{y}$ et $\frac{1}{y}$ pro z' et u), ideoque analytice exprimetur, sicuti problemati convenit.

Sin vero $b_,=o$ fuerit, nec $b_{,,}+c_,=o$, æquatio integranda, factis $\beta z'+c_{,,}=\beta z$ & $\beta=b_{,,}+c_,$, formam $(\alpha z+a)dz=nudz+zdu$ nanciscetur, ideoque ducta in z^{n-1} integrabitur. Ita scil. obtinetur $\int z^{n-1}(\alpha z+a)dz=$ $\int(nuz^{n-1}dz+z^n du)$, seu $\frac{\alpha z^{n+1}}{n+1}+\frac{a z^n}{n}+$ Const. $=z^n u$, indeque $u=$ $+\frac{\alpha z}{n+1}+\frac{a}{n}+C.z^{-n}$. Heic vero scripsimus a pro $\frac{a_{,,}}{\beta}-\frac{a_, c_{,,}}{\beta^2}$, α pro $\frac{a_,}{\beta}$ & n pro $-\frac{b_{,,}}{\beta}$ seu $\frac{-b_{,,}}{c_,+b_{,,}}$. — Sin vero tandem & $b_,=o$ & $b_{,,}+c_,=o$ fuerit, factor erit formæ $\beta^{nz'}$. Quia enim æquatio nostra jam in hanc simpliciorem degeneravit: $(a_,z'+a_{,,})dz'=c_{,,}du+c_,udz'$, erit $n=\frac{c_,}{c_{,,}}$ & $\int(a_,z'+a_{,,})\beta^{nz'}dz'=c_{,,}\int(du+nudz')\beta^{nz'}$, seu $\beta^{nz'}.\left(\frac{a_,z'}{n}+b\right)+$ Const. $=c_{,,}\beta^{nz'}.u$, siquidem fuerit $nb=a_{,,}-\frac{a_,}{n}$.

Quia vero primo casu re vera, ut facile vides, factorem adhibuimus $\frac{u^3}{(z^2-e^2).\overline{z-e}^{\frac{1+n}{2}}.\overline{z+e}^{\frac{1-n}{2}}}=\frac{u^3}{\overline{z-e}^{\frac{3+n}{2}}.\overline{z+e}^{\frac{3-n}{2}}}$ algebraicum, & Cel. LaGrange

demonstravit, hunc atque solutionem particularem communi gaudere factore vel divisore, *alium integrandi* modum tentemus, qui in eo consistet, ut prævie solutionem particularem perquiramus, deindeque factorem idoneum divinemur, quo facto integratio ex methodo notissima perficietur. Hinc

Problema II.

Eandem æquationem aliter integrare, et quidem ita, ut factor ad ipsam integrandam idoneus immediate perquiratur:

Ponamus igitur $y'=\alpha x'+\beta$, quæ solutio particularis sit, et $a_0=o$, eritque $dy'=\alpha dx'$, $y'dx'-x'dy'=\beta dx'$, et, substituendo in data, $(a_{,}x'+a_{,,}(\alpha x'+\beta)).\beta+b_0+b_{,}x'+b_{,,}(\alpha x'+\beta)+(c_0+c_{,}x'+c_{,,}(\alpha x'+\beta)).\alpha=o$, ideoque separatim, x' exsistente arbitrario, $(a_{,}+a_{,,}\alpha)\beta+b_{,}+b_{,,}\alpha+(c_{,}+c_{,,}\alpha)\alpha=o$ et $a_{,,}\beta^2+b_0+b_{,,}\beta+(c_0+c_{,,}\beta)\alpha=o$,

unde α et β tribus modis invenientur; sc. $\beta=-\dfrac{b_{,}+(b_{,,}+c_{,})\alpha+c_{,,}\alpha^2}{a_{,}+a_{,,}\alpha}$, et

$a_{,,}\beta+b_{,,}+c_{,,}\alpha=\dfrac{b_0+c_0\alpha}{-\beta}$, atque inde hoc idem $=\dfrac{(\overline{a_{,}b_{,,}})+(\overline{a_{,}c_{,,}})\alpha}{a_{,}+a_{,,}\alpha}$, et igitur

$(b_0+c_0\alpha).(a_{,}+a_{,,}\alpha)^2=(b_{,}+\overline{b_{,,}+c_{,}}\alpha+c_{,,}\alpha^2).(\overline{a_{,}b_{,,}}+\overline{a_{,}c_{,,}}.\alpha)$, unde invenientur tres radices α_0, $\alpha_{,}$, $\alpha_{,,}$ et simul $\beta=\beta_0$, $\beta_{,}$ et $\beta_{,,}$. Habentur igitur ita tres solutiones particulares $\alpha_0x'+\beta_0-y'$, $\alpha_{,}x'+\beta_{,}-y'$ et $\alpha_{,,}x'+\beta_{,,}-y'$; quare factor idoneus divinetur

$(\alpha_0x'+\beta_0-y')^{n_0}.(\alpha_{,}x'+\beta_{,}-y')^{n_{,}}.(\alpha_{,,}x'+\beta_{,,}-y')^{n_{,,}}=\varphi$. Sin vero jam b_0 et $c_0=o$ facta fuerint, et igitur $x'=x$, $y'=y$, erit α vel

$=-\dfrac{\overline{a_{,}b_{,,}}}{\overline{a_{,}c_{,,}}}=\dfrac{a_{,,}b_{,}-a_{,}b_{,,}}{a_{,}c_{,,}-a_{,,}c_{,}}$, $(=\alpha_0)$ vel $=$ alterutri radicum $(\alpha_{,}, \alpha_{,,})$ æqua-

tionis $o = b_{,} + (b_{,,} + c_{,})\alpha + c_{,,}\alpha^2$, ideoque β vel $= -\frac{b_{,} + (b_{,,} + c_{,}).\alpha_0 + c_{,,}\alpha_0^2}{a_{,} + a_{,,}\alpha_0} = \beta_0$, vel $= o\,(=\beta_{,} = \beta_{,,})$, et igitur $\varphi = (\alpha_0 x + \beta_0 - y)^{n_0}.(\alpha_{,}x - y)^{n_{,}}.(\alpha_{,,}x - y)^{n_{,,}}$ atque $L\varphi = n_0 L\overline{\alpha_0 x + \beta_0 - y} + n_{,}L(\alpha_{,}x - y) + n_{,,}L(\alpha_{,,}x - y)$, et differentiando

$$\frac{\partial_x \varphi}{\varphi} = \frac{n_0\alpha_0}{\alpha_0 x + \beta_0 - y} + \frac{n_{,}\alpha_{,}}{\alpha_{,}x - y} + \frac{n_{,,}\alpha_{,,}}{\alpha_{,,}x - y} \text{ atque}$$

$$\frac{-\partial_y \varphi}{\varphi} = \frac{n_0}{\alpha_0 x + \beta_0 - y} + \frac{n_{,}}{\alpha_{,}x - y} + \frac{n_{,,}}{\alpha_{,,}x - y}.$$ Si vero fuerit $\varphi(Xdx + Ydy)$ differentiale exactum, erit $\partial_y(\varphi X) = \partial_x(\varphi Y)$, seu $X.\frac{\partial_y \varphi}{\varphi} + \partial_y X = Y.\frac{\partial_x \varphi}{\varphi} + \partial_x Y$, hoc est $Y\partial_x L\varphi - X\partial_y L\varphi = \partial_y X - \partial_x Y$. In præsenti vero casu est $X = (a_{,}x + a_{,,}y)y + B'$ atque $Y = C' - (a_{,}x + a_{,,}y)x$, ideoque $\partial_y X = a_{,}x + 2a_{,,}y + b_{,,}$ atque $\partial_x Y = c_{,} - 2a_{,}x - a_{,,}y$, et igitur æquatio factorem definiens $Y\partial_x L\varphi - X\partial_y L\varphi = 3a_{,}x + 3a_{,,}y + b_{,,} - c_{,}$, seu

$$\left(C' - \overline{a_{,}x + a_{,,}y}\,x\right)\cdot\left(\frac{n_0\alpha_0}{\alpha_0 x + \beta_0 - y} + \frac{n_{,}\alpha_{,}}{\alpha_{,}x - y} + \frac{n_{,,}\alpha_{,,}}{\alpha_{,,}x - y}\right) +$$
$$+ (B' + (a_{,}x + a_{,,}y)y)\cdot\left(\frac{n_0}{\alpha_0 x + \beta_0 - y} + \frac{n_{,}}{\alpha_{,}x - y} + \frac{n_{,,}}{\alpha_{,,}x - y}\right)$$
$$= 3(a_{,}x + a_{,,}y) + b_{,,} - c_{,}.$$ Hæc vero æquatio, ponendo $n_0 + n_{,} + n_{,,} + 3 = o$, quo omnes quarti ordinis termini tollantur, in hanc migrat tertii ordinis:

$$n_0(a_{,}x + a_{,,}y)\beta_0.(b_{,}x^2 + \overline{b_{,,} + c_{,}}xy + c_{,,}y^2) + C'n_0\alpha_0(b_{,}x^2 + \overline{b_{,,} + c_{,}}xy + c_{,,}y^2) +$$
$$+ C'c_{,,}(\beta_0 + \alpha_0 x - y)(n_{,}\alpha_{,}\overline{\alpha_{,,}x - y} + n_{,,}\alpha_{,,}\overline{\alpha_{,}x - y}) +$$
$$+ B'[n_0(b_{,}x^2 + \overline{b_{,,} + c_{,}}xy + c_{,,}y^2) + \overline{\alpha_0 x + \beta_0 - y}.(n_{,}\overline{\alpha_{,,}x - y} + n_{,,}\overline{\alpha_{,}x - y})c_{,,}]$$
$$= (b_{,,} - c_{,}).(\beta_0 + \alpha_0 x - y).(b_{,}x^2 + \overline{b_{,,} + c_{,}}xy + c_{,,}y^2),$$ ubi $C' = c_{,}x + c_{,,}y$ et $B' = b_{,}x + b_{,,}y$. Ut vero huic æquationi satisfiat, necessario sit, (nisi $\beta_0 = o$)

1) $C'c_{,,}(\alpha_{,}n_{,}\overline{\alpha_{,,}x-y}+n_{,,}\alpha_{,,}\overline{\alpha_{,}x-y})+B'(n_{,}\overline{\alpha_{,,}x-y}+n_{,,}\overline{\alpha_{,}x-y})c_{,,}$
$=(b_{,,}-c_{,}).(b_{,}x^2+\overline{b_{,,}+c_{,}}xy+c_{,,}y^2)$, ideoque primo (ex terminis x^2 et y^2 tenentibus) $c_{,}c_{,,}(n_{,,}\alpha_{,,}\alpha_{,}+n_{,}\alpha_{,}\alpha_{,,})+b_{,}(n_{,}\alpha_{,,}+n_{,,}\alpha_{,})c_{,,}=(b_{,,}-c_{,})b_{,}$ et
$c_{,,}^2(n_{,}\alpha_{,}+n_{,,}\alpha_{,,})+b_{,,}(n_{,}+n_{,,})c_{,,}=(c_{,}-b_{,,})c_{,,}$, seu
$(c_{,}\alpha_{,}+b_{,})\alpha_{,,}.n_{,}+(c_{,}\alpha_{,,}+b_{,})\alpha_{,}.n_{,,}=(b_{,,}-c_{,})b_{,}:c_{,,}$
et $(c_{,,}\alpha_{,}+b_{,,})n_{,}+(c_{,,}\alpha_{,,}+b_{,,})n_{,,}=(c_{,}-b_{,,})$, quæ ambæ tamen æquationes sibi conveniunt, quia $b_{,}+c_{,}\alpha=-(b_{,,}+c_{,,}\alpha)\alpha$ et $\alpha_{,}\alpha_{,,}c_{,,}=b_{,}$. Ex coefficiente ipsius xy deinde invenitur
$(\overline{n_{,}+n_{,,}}.\alpha_{,}\alpha_{,,}c_{,,}-\overline{n_{,}\alpha_{,}+n_{,,}\alpha_{,,}}c_{,}+\overline{n_{,}.\alpha_{,,}+n_{,,}\alpha_{,}}b_{,,}-b_{,}\overline{n_{,}+n_{,,}}).c_{,,}=b_{,,}^2-c_{,}^2$, seu
$b_{,}$ exsistente $\alpha_{,}\alpha_{,,}c_{,,}$, $\overline{b_{,,}\alpha_{,,}-c_{,}\alpha_{,}}n_{,}+\overline{b_{,,}\alpha_{,}-c_{,}\alpha_{,,}}n_{,,}=(b_{,,}^2-c_{,}^2):c_{,,}$, ex qua æquatione unacum alterutra præcedentium $n_{,}$ et $n_{,,}$ determinari videntur.
2) Deinde vero superest æquatio trium dimensionum (seu homogenea tertii gradus formæ $3^c xy=o$): $n_0\beta_0(a_{,}x+a_{,,}y).(b_{,}x^2+\overline{b_{,,}+c_{,}}xy+c_{,,}y^2)+$
$+(c_{,}x+c_{,,}y).[n_0\alpha_0(b_{,}x^2+\overline{b_{,,}+c_{,}}xy+c_{,,}y^2)+\overline{\alpha_0x-y}c_{,,}(n_{,}\alpha_{,}.\overline{\alpha_{,,}x-y}+n_{,,}\alpha_{,,}\overline{\alpha_{,}x-y})]$
$+(b_{,}x+b_{,,}y).[n_0(b_{,}x^2+\overline{b_{,,}+c_{,}}xy+c_{,,}y^2)+\overline{\alpha_0x-y}c_{,,}(n_{,}\overline{\alpha_{,,}x-y}+n_{,,}\overline{\alpha_{,}x-y})]$
$=(b_{,,}-c_{,})(\alpha_0x-y).(b_{,}x^2+\overline{b_{,,}+c_{,}}xy+c_{,,}y^2)$, quæ jam identica sit, cum plures numeri arbitrio haud relinquantur. Quod ut comprobetur, ponamus

a) $y=\alpha_0 x$, eritque

$$\left.\begin{array}{l} n_0\beta_0(a_{,}+a_{,,}\alpha_0).(b_{,}+\overline{b_{,,}+c_{,}}\alpha_0+c_{,,}\alpha_0^2)\\ n_0.[b_{,}+b_{,,}\alpha_0+(c_{,}+c_{,,}\alpha_0)\alpha_0].(b_{,}+\overline{b_{,,}+c_{,}}\alpha_0+c_{,,}\alpha_0^2)\end{array}\right\}=o$$

seu $n_0(b_{,}+\overline{b_{,,}+c_{,}}\alpha_0+c_{,,}\alpha_0^2)[b_{,}+\overline{b_{,,}+c_{,}}\alpha_0+c_{,,}\alpha_0^2+(a_{,}+a_{,,}\alpha_0)\beta_0]=o$, id quod ad valorem ipsius β_0 jam inventum bene quadrat.

b) $y=\alpha x$ (exsistente $\alpha=\alpha_{,}$ vel $=\alpha_{,,}$, ideoque $b_{,}+\overline{b_{,,}+c_{,}}\alpha+c_{,,}\alpha^2=o$, et igitur $b_{,}x^2+\overline{b_{,,}+c_{,}}xy+c_{,}y^2=o$), sitque necessario
$c_{,,}[(c_{,}+c_{,,}\alpha).(\alpha_{,}(\alpha_{,,}-\alpha)n_{,}+\alpha_{,,}\overline{\alpha_{,}-\alpha}n_{,,})+\overline{b_{,}+b_{,,}\alpha}.(n_{,}\overline{\alpha_{,,}-\alpha}+n_{,,}\overline{\alpha_{,}-\alpha})]\overline{\alpha_0-\alpha}=o$

seu 1) quando $\alpha=\alpha_{,}$ fuerit, (nec $\alpha_0=\alpha_{,}$),
$[\overline{c_{,}+c_{,,}\alpha_{,}}.(n_{,}\alpha_{,}\overline{\alpha_{,,}-\alpha_{,}})+(b_{,}+b_{,,}\alpha_{,}).n_{,}\overline{\alpha_{,,}-\alpha_{,}}]c_{,,}=o$
seu $\overline{\alpha_{,,}-\alpha_{,}}.n_{,}.(\overline{c_{,}+c_{,,}\alpha_{,}}\alpha_{,}+b_{,}+b_{,,}\alpha_{,})\,c_{,,}=o$; similiterque 2) cum $\alpha=\alpha_{,,}$, $\overline{\alpha_{,}-\alpha_{,,}}.n_{,,}.(\overline{c_{,}+c_{,,}\alpha_{,,}}.\alpha_{,,}+b_{,}+b_{,,}\alpha_{,,})=o$, id, quod utrumque æquationi ipsius α convenit.

c) Sit tandem $x=o$, (seu $y=\infty$), et inde
$n_0(\beta_0 a_{,,}c_{,,}+c_{,,}^2\alpha_0)+c_{,,}^2(n_{,}\alpha_{,}+n_{,,}\alpha_{,,})+b_{,,}(n_0 c_{,,}+c_{,,}(n_{,}+n_{,,}))=-\overline{b_{,,}-c_{,}}c_{,,}$, seu
$n_0 c_{,,}(\beta_0 a_{,,}+c_{,,}\alpha_0+b_{,,})=o$, quia jam $\overline{b_{,,}+c_{,,}\alpha_{,}}n_{,}+\overline{b_{,,}+c_{,,}\alpha_{,,}}n_{,,}=c_{,}-b_{,,}$.

Invenimus vero $\alpha_0=\dfrac{\overline{a_{,,}b_{,}}}{\overline{a_{,}c_{,,}}}$ atque $-\beta_0=\dfrac{b_{,}+\overline{b_{,,}+c_{,}}\alpha_0+c_{,,}\alpha_0^2}{a_{,}+a_{,,}\alpha_0}$

seu $=\dfrac{\overline{b_{,}c_{,,}}}{\overline{a_{,}c_{,,}}}$, hoc scil. valore rite substituto atque factore communi $a_{,}^2 c_{,,}-a_{,}a_{,,}\overline{c_{,}+b_{,,}}+a_{,,}^2 b_{,}$ suppresso; (sin vero hic evanuerit, β_0 hinc determinaretur,): ideoque hæc æquatio $\beta_0 a_{,,}+c_{,,}\alpha_0+b_{,,}=o$ in hanc migraret $c_{,,}.\overline{a_{,,}b_{,}}-a_{,,}.\overline{b_{,}c_{,,}}+b_{,,}.\overline{a_{,}c_{,,}}=o$, seu $\overline{a_{,}b_{,,}c_{,,}}=o$, quæ quidem bene sibi constat. Jam igitur æquatio illa integra tertii ordinis pro quatuor diversis ipsius $y:x$ valoribus (α_0, $\alpha_{,}$, $\alpha_{,,}$ et ∞) valet, ideoque pro quibusvis valebit, seu identica erit.

Est igitur φ seu $\overline{\alpha_0 x+\beta_0-y}^{\,n_0}.\overline{\alpha_{,}x-y}^{\,n_{,}}\overline{\alpha_{,,}x-y}^{\,n_{,,}}$ factor ad æquationem $(a_{,}x+a_{,,}y)(xdy-ydx)-(b_{,}x+b_{,,}y)dx-(c_{,}x+c_{,,}y)dy=o$ integrandam idoneus, dummodo

1) $\alpha_0.\overline{a_{,}c_{,,}}+\overline{a_{,}b_{,,}}=o$, 2) $\overline{a_{,}c_{,,}}.\beta_0+\overline{b_{,}c_{,,}}=o$, 3) et 4) $2^0\alpha_{,}=o=2^0\alpha_{,,}$,
(exsistente $2^0\alpha=b_{,}+\overline{b_{,,}+c_{,}}\alpha+c_{,,}\alpha^2$,)
5) $(c_{,,}\alpha_{,}+b_{,,})n_{,}+(c_{,,}\alpha_{,,}+b_{,,})n_{,,}=(c_{,}-b_{,,})$
6) $(b_{,,}\alpha_{,,}-c_{,}\alpha_{,})n_{,}+(b_{,,}\alpha_{,}-c_{,}\alpha_{,,})n_{,,}=(b_{,,}^2-c_{,}^2):c_{,,}$,
& 7) $n_0+n_{,}+n_{,,}+3=o$ fuerint. Quarum quidem æquationum 1) 2) 3)

& 4) tres α cum β_0 determinant, 5) vero et 6) $n_{,}$ et $n_{,,}$ definirent; quo facto n_0 ex 7) facile petitur. Quia vero 5) & 6) ejus est formæ, quæ interdum indefinita vel etiam contraria evadere potest; num quantitas s. d. determinans (vel potius *limitans* vel *definiens*) evanescat, videndum est. Hic vero $(b_{,,}+c_{,,}\alpha_{,})(b_{,,}\alpha_{,}-c_{,}\alpha_{,,})+(b_{,,}+c_{,,}\alpha_{,,})(c_{,}\alpha_{,}-b_{,,}\alpha_{,,})$ evadit $=$ $(\alpha_{,}-\alpha_{,,}).b_{,,}(b_{,,}+c_{,}+c_{,,}\overbrace{\alpha_{,}+\alpha_{,,}})$, quod ex indole functionis $2^o\alpha$ evanescit. Alteruter igitur casus timendus: et quidem si æquationes 5) et 6) sibi adversarentur, tota res rueret. Valoribus vero $c_{,,}\alpha_{,}\alpha_{,,}$ et $c_{,}+c_{,,}(\alpha_{,}+\alpha_{,,})$ in locum ipsorum $b_{,}$ et $-b_{,,}$ suffectis, æquatio 5) hanc induit formam: $\overbrace{c_{,}+c_{,,}\alpha_{,,}}n_{,}+\overbrace{c_{,}+c_{,,}\alpha_{,}}n_{,,}+2c_{,}+c_{,,}\overbrace{\alpha_{,}+\alpha_{,,}}=o$, atque æquatio 6), factore $\alpha_{,}+\alpha_{,,}$ sejuncto, eandem omnino adsciscit formam, ideoque ambæ illæ æquationes in hanc unam coalescunt, quare alterutra quantitatum $n_{,}$ et $n_{,,}$ arbitrio relinquitur. Factore vero φ feliciter ita invento, integratio modo notissimo perficietur; quæ tamen functiones magis transcendentis formæ $\int(B+Cy)dy(A_0-y)^{n_0}(A_1-y)^{n_{,}}(A_2-y)^{n_{,,}}$ postulare videtur. Antiquum igitur deseramus, simpliciorem quærentes. Quem quidem elegantibus his *Theorematibus* superstruamus.

Theorema I.

Si ad æquationem differentialem primi Ordinis $Xdx+Ydy=o$ *duo vel plura inventa sunt integralia* ($\Phi=$ const., $\Psi=$ const.) *completa, pars variabilis altera alterius necessario sit functio quædam* (F) (ut $\Psi=F\Phi$). *)

Differentiando enim erit $\Phi' dx+\Phi_{,}dy=o$ et $\Psi' dx+\Psi_{,}dy=o$, ideoque $\Phi':\Phi_{,}=-dy:dx=X:Y=\Psi':\Psi_{,}$ seu $\partial_x\Phi:\partial_y\Phi=\partial_x\Psi:\partial_y\Psi$, cujus quidem æqua-

*) Simile Theorema de Integralibus æquationum altiorum valet, de quo alio loco seorsim agemus, idque eo magis, quod in libris nonnullis elementaribus hac de re erroneum quid pro vero venditatur.

tionis particularis solutio est $\Psi = F(\Phi)$, eaque completa, exsistente F functione arbitraria. Hac enim æquatione admissa, erit $\partial_x \Psi = \partial_x \Phi . F_{\prime} \Phi$ et $\partial_y \Psi = \partial_y \Phi . F_{\prime} \Phi$, unde ratio inventa mox liquet.

Hinc vero facile colliges, transcendentem, in secundo modo sese offerentem, ut functionem illius in primo inventæ considerari posse: quæ res haud spernendam harum transcendentium comparationem subministrat, eamque peculiari tractatu dignam.

Theorema II.

Duobus æquationis cujusdam differentialis $Xdx + Ydy = o$ inventis factoribus φ et ψ, integrationi inservientibus, erit $\psi = \varphi . C$ illius integrale completum, (utut constante arbitrario C gaudens).

Quia enim formula $Xdx + Ydy$ in φ vel ψ ducta differentiale fit exactum, sit tum $\int \varphi (Xdx + Ydy) = \Phi$
tum $\int \psi (Xdx + Ydy) = \Psi$, exsistentibus Φ et Ψ functionibus ipsorum x et y hac ipsa integratione inventis. Quæ cum ita sint, propositæ æquationis integrale erit tum $\Phi =$ const. tum $\Psi =$ const.; nec vero, ut modo demonstravimus, huic eidem æquationi $Xdx + Ydy = o$ ambo hæc integralia competere possunt, nisi fuerit $\Psi = F(\Phi) =$ functio aliqua ipsius Φ, ideoque $d\Psi = F_{\prime}\Phi . d\Phi$. Est vero jam universim $d\Psi = \psi (Xdx + Ydy)$ atque $d\Phi = \varphi (Xdx + Ydy)$, quamobrem necessario $\psi = \varphi . F_{\prime}\Phi$ sit. Admisso igitur integrale $\Phi =$ const., erit sane $F_{\prime}\Phi =$ const. $= C$, ideoque $\psi = \varphi . C$ aliud integrale, idque æque completum.

Sic quoniam in casu nostro factorem invenimus φ formæ $z_0^{n0} . z_1^{n1} . z_2^{n2}$, exsistente singulo z formæ $= \beta + \alpha x' - y'$ atque alterutro (ex. gr. n_2) exponentium arbitrario, novum quidem factorem ψ inveniemus, alium huic tribuendo valorem (n_2^1), nempe $\psi = z_0^{n_0^1} . z_1^{n_1^1} . z_2^{n_2^1}$, quare ex hoc ipso theoremate integrale completum erit $(\psi : \varphi =) \, z_0^{n_0^1 - n_0} . z_1^{n_1^1 - n_1} . z_2^{n_2^1 - n_2} =$ Const.

Jam vero facto $n+1=r$, exponentium relationes inveniuntur $r_0+r_1+r_2=o$ atque $r_,+\mu.r_{,,}=o$, si $\mu=c_,+c_{,,}\alpha_,:c_,+c_{,,}\alpha_{,,}$. Sumto igitur primum $r_{,,}=o$, erit $r_,=o=r_0$, ideoque quique $n^1=-1$, et $n^1-n=-(n+1)=-r$ et factor $\psi=(z_0.z_1.z_2)^{-1}$, quare integrale $z_0^{r_0}.z_1^{r_1}.z_2^{r_2}=$ Const. seu $(\beta_0+\alpha_0 x-y)^{r_0}.(\alpha_, x-y)^{r_,}.(\alpha_{,,}x-y)^{r_{,,}}=$ Const.

Cum adeo integrale satis simplex algebraicum invenimus; videamus, numne primum saltem integrale ejusmodi evadat, deindeque an simile quid tertio quodam magis directo modo efficere valeamus.

Ad formulam igitur primo modo inventam $\dfrac{(\alpha z+a)dz}{(z-e)^{\frac{3+n}{2}}.(z+e)^{\frac{3-n}{2}}}$ integrandam, ponamus $\alpha z+a=\gamma\overline{z-e}+\beta.\overline{z+e}$, seu $\alpha=\beta+\gamma$, $a=e(\beta-\gamma)$, eritque $\beta=\dfrac{1}{2}\left(\alpha+\dfrac{a}{e}\right)$, $\gamma=\dfrac{1}{2}\left(\alpha-\dfrac{a}{e}\right)$, et

$$\frac{u}{Z}=\gamma.\int\frac{dz}{\overline{z-e}^{\frac{1+n}{2}}.\overline{z+e}^{\frac{3-n}{2}}}+\beta.\int\frac{dz}{\overline{z-e}^{\frac{3+n}{2}}.\overline{z+e}^{\frac{1-n}{2}}}$$

$$=\gamma.\int\frac{dz}{\overline{z+e}^2.\left(\frac{z-e}{z+e}\right)^{\frac{1+n}{2}}}+\beta.\int\frac{dz}{\overline{z-e}^2.\left(\frac{z+e}{z-e}\right)^{\frac{1-n}{2}}}.$$

Ponamus ulterius 1) $z+e=\dfrac{1}{\zeta}$, unde $z-e=\dfrac{1}{\zeta}-2e$, $\dfrac{z-e}{z+e}=1-2e\zeta$, et $\dfrac{dz}{(z+e)^2}=-d\zeta$, ideoque

$$\text{pars prima}=-\gamma\int\frac{d\zeta}{(1-2e\zeta)^{\frac{1+n}{2}}}=-\gamma\int d\zeta(1-2e\zeta)^{-\frac{1+n}{2}}=\gamma\frac{\overline{1-2e\zeta}^{\frac{1-n}{2}}}{e(1-n)},$$

seu $=\dfrac{\gamma}{e(1-n)}\cdot\left(\dfrac{z-e}{z+e}\right)^{\frac{1-n}{2}}$ evadit. Sit deinde 2)

$z-e=x^{-1}$, unde $z+e=x^{-1}+2e$, $\dfrac{z+e}{z-e}=1+2ex$, $\dfrac{dz}{(z-e)^2}=-dx$;

erit igitur pars altera =

$$=-\beta . \int dx . (1+2ex)^{\frac{n-1}{2}} = -\frac{\beta}{e} \cdot \frac{(1+2ex)^{\frac{n+1}{2}}}{1+n} = -\frac{\beta}{e(1+n)} \cdot \left(\frac{z+e}{z-e}\right)^{\frac{1+n}{2}},$$

quare $$\frac{u}{\overline{z-e}^{\frac{1+n}{2}} \cdot \overline{z+e}^{\frac{1-n}{2}}} = \frac{\gamma}{e(1-n)} \cdot \left(\frac{z-e}{z+e}\right)^{\frac{1-n}{2}} - \frac{\beta}{e(1+n)} \cdot \left(\frac{z+e}{z-e}\right)^{\frac{1+n}{2}} + \text{Cst.}$$

Quoniam vero, docente Theor. I., theoriæ functionum maxime interest, ut quæque æquatio diversimode integretur; videamus, num etiam

Problema III.

Tertio modo æquatio nostra secundi gradus integretur. Qnæ quidem ad hanc primi revocatur, (jam diu integratam) $(1_0+1_, x'+1_{,,} y').dx'+(1_0+1_, x'+1_{,,} y') . dy' = o$. Ponatur enim $x'=\frac{p}{r}$ et $y'=\frac{q}{r}$, scribaturque $1^o rpq$ pro $1_0 . r+1_, p+1_{,,} q$ (et similiter $1^o rpq$ pro $1_0 r+1_, p+1_{,,} q$); eritque $1^o rpq . (rdp-pdr)+1^o rpq . (rdq-qdr) = o$. Sit vero jam $p=p_0+p_, x+p_{,,} y$, et similiter $q=q_0+q_, x+q_{,,} y$ et $r=r_0+r_, x+r_{,,} y$, eritque $1^o \overline{rpq} = 1^o r_0 p_0 q_0 + x . 1^o r_, p_, q_, + y 1^o r_{,,} p_{,,} q_{,,}$ seu breviter $1^o \overline{rpq} = 1_0^0 + x . 1_,^0 + y . 1_{,,}^0$, (ubi $1_,^0 = 1^o r_, p_, q_, = 1_0 r_, + 1_, p_, + 1_{,,} q_,$, et similiter in reliquis), atque $r\underset{x}{\partial} p - p\underset{x}{\partial} r = rp_, - pr_, = \widehat{q}_{,,} - \widehat{q}_0 . y$, (ubi $\widehat{q}_{,,}$ vel $\widehat{q}_{,,} = r_0 p_, - r_, p_0 = \overline{r_0 p_,}$, et $\widehat{q}_0 = \overline{r_, p_{,,}}$), similiterque $r\underset{x}{\partial} q - q\underset{x}{\partial} r = \widehat{p}_{,,} - \widehat{p}_0 . y$, atque $r\underset{y}{\partial} p - p\underset{y}{\partial} r = \widehat{q}_, + \widehat{q}_0 . x$, et $r\underset{y}{\partial} q - q\underset{y}{\partial} r = \overline{p}_, + \widehat{p}_0 . x$, (ubi simili sensu $\widehat{q}_, = \overline{r_0 p_{,,}}$, $\widehat{q}_0 = \overline{r_, p_{,,}}$, $\overline{p}_, = \overline{r_0 q_{,,}}$, $\overline{p}_0 = \overline{r_, q_{,,}}$). His igitur valoribus substitutis, nostra æquatio integrabilis (seu ad primum gradum revocabilis) erit

$\left(1^{0} rpq.(\overline{q}_{,,}-\overline{q}_{0}.y)+1^{0} rpq.(\overline{p}_{,,}-\overline{p}_{0} y)\right) dx+$

$+\left(1^{0} rpq.(\overset{\frown}{q}_{,}+\overset{\frown}{q}_{0}.x)+1^{0} rpq.(\overset{\frown}{p}_{,}+\overset{\frown}{p}_{0}.x)\right).dy=o$ seu

$\left(1^{0} rpq.\overset{\frown}{q}_{,,}+1^{0} rpq.\overset{\frown}{p}_{,,}-(1^{0} rpq.\overset{\frown}{q}_{0}+1^{0} rpq.\overline{p}_{0}).y\right) dx+$

$+\left(1^{0} rpq.\overset{\frown}{q}_{,}+1^{0} rpq.\overset{\frown}{p}_{,}+(1^{0} rpq.\overset{\frown}{q}_{0}+1^{0} rpq.\overline{p}_{0}).x\right).dy=o$, ideoque aperte formæ $(a_0+a_{,}x+a_{,,}y)(xdy-ydx)+(b_0+b_{,}x+b_{,,}y)dx+(c_0+c_{,}x+c_{,,}y)dy=o$ ab initio propositæ. Comparatione igitur rite instituta, novem exorientur æquationes, ad quantitates arbitrarias 1_0, $1_{,}$, $1_{,,}$, 1_0, $1_{,}$, $1_{,,}$, p_0, $p_{,}$, $p_{,,}$, q_0, $q_{,}$, $q_{,,}$, r_0, $r_{,}$, $r_{,,}$ determinandas inservientes. Quoniam vero harum numerus (15) multo major est, simul aliis desideriis satisfacere licet. Ponamus igitur ipsas a_0, b_0 et c_0 jam $=o$ esse factas, et efficiamus ut $1_0=o=1$, et $1_0=o=1_{,,}$ sint, seu æquatio reducta hujus simplicioris formæ $1_{,,}.y'dx'+1_{,}.x'dy'=o$. Quia igitur $a_0=o=b_0=c_0$, erit

$1^0_0.\overset{\frown}{q}_0+1^0_0.\overset{\frown}{p}_0=o$, $1^0_0.\overset{\frown}{q}_{,,}+1^0_0.\overset{\frown}{p}_{,,}=o=1^0_0.\overset{\frown}{q}_{,}+1^0_0.\overset{\frown}{p}_{,}$, unde colligitur $1^0_0=o=1^0_0$ seu $1_0 r_0+1_{,}p_0+1_{,,}.q_0=o=1_0.r_0+1_{,}p_0+1_{,,}.q_0=o$, h. e. $1_{,,}.q_0=o=1_{,}.p_0$, et igitur $q_0=o=p_0$, qui valores tribus æquationibus satisfaciunt. Reliquæ rationes sunt:

$c_{,}:c_{,,}:b_{,}:b_{,,}:a_{,}:a_{,,}=1^0_{,}.\overset{\frown}{q}_{,}+1^0_{,}.\overset{\frown}{p}_{,}:1^0_{,,}.\overset{\frown}{q}_{,}+1^0_{,,}.\overset{\frown}{p}_{,}:1^0_{,}.\overset{\frown}{q}_{,,}+1^0_{,}.\overset{\frown}{p}_{,,}:$

$:1^0_{,,}.\overset{\frown}{q}_{,,}+1_{,}^0.\overset{\frown}{p}_{,,}:1_{,}^0.\overset{\frown}{q}_0+1_{,}^0.\overset{\frown}{p}_0:1^0_{,,}.\overset{\frown}{q}_0+1_{,,}^0.\overset{\frown}{p}_0$, quarum illæ $p_{,}:p_{,,}:q_{,}:q_{,,}$ cum $1_{,,}:1$, et hæ $r_0:r_{,}:r_{,,}$ suppeditant. Obtinemus scilicet $\overset{\frown}{p}_{,}:\overset{\frown}{q}_{,}=c_{,,}1^0_{,}-c_{,}1^0_{,,}:$ $:c_{,}1_{,,}^0-c_{,,}1^0_{,}$, et $\overset{\frown}{p}_{,,}:\overset{\frown}{q}_{,,}=b_{,,}1_{,}^0-b_{,}1^0_{,,}:b_{,}1^0_{,,}-b_{,,}1_{,}^0$, atque $\overset{\frown}{p}_0:\overset{\frown}{q}_0=$ $=a_{,,}1_{,}^0-a_{,}1_{,,}^0:a_{,}1^0_{,,}-a_{,,}1_{,}^0$. Quia vero jam $p_0=o=q_0$, erit $\overset{\frown}{q}_{,,}=r_0 p_{,}$, $\overset{\frown}{p}_{,,}=r_0 q_{,}$, $\overset{\frown}{q}_{,}=r_0 p_{,,}$, $\overset{\frown}{p}_{,}=r_0 q_{,,}$, et igitur $\overset{\frown}{q}_{,,}:\overset{\frown}{p}_{,,}=p_{,}:q_{,}$ atque $\overset{\frown}{p}_{,}:\overset{\frown}{q}_{,}=q_{,,}:p_{,,}$. Est vero præterea $1_{,}^0=1_{,,}q_{,}$, $1^0_{,,}=1_{,,}q_{,,}$, $1^0_{,}=1_{,}p_{,}$, $1_{,,}^0=1_{,}.p_{,,}$; quare $q_{,,}:p_{,,}=(c_{,,}q_{,,}-c_{,}q_{,,})1_{,,}:(c_{,}p_{,,}-c_{,,}p_{,})1$, et $q_{,}:p_{,}=\overset{\frown}{p}_{,,}:\overset{\frown}{q}_{,,}=(b_{,,}q_{,}-b_{,}q_{,,})1_{,,}:(b_{,}p_{,,}-b_{,,}p_{,}).1_{,}$, unde

$$q_{,,}p_{,}:q_{,}p_{,,} = \overline{c_{,,}q_{,} - c_{,}q_{,,}} . \overline{b_{,}p_{,,} - b_{,,}p_{,}} : \overline{c_{,}p_{,,} - c_{,,}p_{,}} . \overline{b_{,,}q_{,} - b_{,}q_{,,}}$$ seu

$$\overline{c_{,,}q_{,} - c_{,}q_{,,}} . \overline{b_{,}p_{,,} - b_{,,}p_{,}} . q_{,}p_{,,} = \overline{c_{,}p_{,,} - c_{,,}p_{,}} . \overline{b_{,,}q_{,} - b_{,}q_{,,}} . q_{,,}p_{,}$$ h. e.

$$c_{,,}b_{,} . (q_{,}^2 . p_{,,}^2 - p_{,}^2 . q_{,,}^2) - c_{,,}b_{,,} . (q_{,}^2 p_{,}p_{,,} - p_{,}^2 q_{,}q_{,,}) =$$
$$= c_{,}b_{,} (p_{,,}^2 q_{,}q_{,,} - p_{,}p_{,,}q_{,,}^2),$$ seu

$$c_{,,}b_{,} (q_{,}p_{,,} + p_{,}q_{,,}) . (q_{,}p_{,,} - p_{,}q_{,,}) - c_{,,}b_{,,}p_{,}q_{,} . (q_{,}p_{,,} - p_{,}q_{,,})$$
$$- c_{,}b_{,}p_{,,}q_{,,} (p_{,,}q_{,} - p_{,}q_{,,}) = o =$$

$$= (q_{,}p_{,,} - p_{,}q_{,,}) . (c_{,,}b_{,}\overline{q_{,}p_{,,} + p_{,}q_{,,}} - c_{,,}b_{,,}p_{,}q_{,} - c_{,}b_{,}p_{,,}q_{,,}) = o.$$ Quia vero $p_{,}:q_{,}$ non $= p_{,,}:q_{,,}$, (quo casu $x':y' = p_{,}:q_{,}$ foret), necessario erit

$$c_{,}b_{,} p_{,,} q_{,,} - c_{,,}b_{,}(q_{,}p_{,,} + p_{,}q_{,,}) + c_{,,}b_{,,}p_{,}q_{,} = o$$

seu $c_{,}b_{,}\pi\varkappa - c_{,,}b_{,}(\pi + \varkappa) + c_{,,}b_{,,} = o$, si $p_{,,} = \pi p_{,}$ et $q_{,,} = \varkappa q_{,}$. Ex rationibus vero modo propositis præterea obtinemus $1_{,}^0(c_{,} . \widehat{q_{,,}} - b_{,} . \widehat{q_{,}}) = 1_{,}^0(\widehat{p_{,}} . b_{,} - \widehat{p_{,,}} c_{,})$ et $1_{,}^0(c_{,} . \widehat{q_0} - a_{,} . \widehat{q_{,}}) = 1_{,}^0(a_{,} . \widehat{p_{,}} - c_{,}\widehat{p_0})$, seu

$$1_{,,} . q_{,} . (c_{,} . r_0 p_{,} - b_{,} . r_0 p_{,,}) = 1_{,}p_{,} . (b_{,} . r_0 q_{,,} - c_{,} . r_0 q_{,})$$ et

$$1_{,,} . q_{,} . (c_{,} . \overline{r_{,}p_{,,}} - a_{,} . r_0 p_{,,}) = - 1_{,}p_{,} . (c_{,} . \overline{r_{,}q_{,,}} - a_{,} . r_0 q_{,,}),$$ unde

$1_{,,} : 1_{,} = p_{,} (b_{,}q_{,,} - c_{,}q_{,}) : q_{,} (c_{,}p_{,} - b_{,}p_{,,})$, quod et modo invenimus $= q_{,}(b_{,}p_{,,} - b_{,,}p_{,}) : p_{,} (b_{,,}q_{,} - b_{,}q_{,,})$, ideoque erit

$$p_{,}^2 (b_{,}q_{,,} - c_{,}q_{,}) (b_{,,}q_{,} - b_{,}q_{,,}) = q_{,}^2 (c_{,}p_{,} - b_{,}p_{,,}) (b_{,}p_{,,} - b_{,,}p_{,})$$ seu

$(b_{,}\varkappa - c_{,}) (b_{,,} - b_{,}\varkappa) = (c_{,} - b_{,}\pi) . (b_{,}\pi - b_{,,})$. Est vero jam

$$\pi = \frac{c_{,,}(b_{,}\varkappa - b_{,,})}{b_{,} . (c_{,}\varkappa - c_{,,})}, \quad b_{,}\pi - b_{,,} = \frac{\overline{c_{,,}b_{,}} . \varkappa}{c\varkappa - c_{,,}}, \quad b_{,}\pi - c_{,} = \frac{(c_{,,}b_{,} - c_{,}^2)\varkappa + c_{,,}\overline{c_{,} - b_{,,}}}{c_{,}\varkappa - c_{,,}},$$ et

igitur $(b_{,}\varkappa - c_{,}) (b_{,,} - b_{,}\varkappa) . \overline{c_{,}\varkappa - c_{,,}}^2 + \overline{c_{,,}b_{,}} . \varkappa (\overline{c_{,,}b_{,} - c_{,}}^2 \varkappa + c_{,,}\overline{c_{,} - b_{,,}}) = o$. Quare $\varkappa$ universim quatuor diversis modis determinatur, et exinde π, atque $1_{,,} : 1_{,} = (b_{,}\varkappa - c_{,}) : (c_{,} - b_{,}\pi)$, quod ponatur $= n$.

Tandem ex $\overline{r_{,}q_{,,}} : \overline{r_{,}p_{,,}} = (a_{,,} . q_{,} - a_{,}q_{,,}) . 1_{,,} : (a_{,}p_{,,} - a_{,,}p_{,}) 1_{,}$ et $1_{,,}q_{,} (c_{,} . \overline{r_{,}p_{,,}} - r_0 . a_{,}p_{,,}) = 1_{,}p_{,} (r_0 . a_{,}q_{,,} - c_{,} . \overline{r_{,}q_{,,}})$ simul revera inventis, obtinentur $\overline{r_{,}q_{,,}}$ et $\overline{r_{,}p_{,,}}$, nempe $\overline{r_{,}p_{,,}} = \dfrac{\overline{r_{,}q_{,,}} . (a_{,}p_{,,} - a_{,,}p_{,}) 1_{,}}{1_{,,} . (a_{,,}q_{,} - a_{,}q_{,,})}$, et igitur

$$c_{,}\left(q_{,}\cdot\frac{(a_{,}p_{,,}-a_{,,}p_{,})}{a_{,,}q_{,}-a_{,}q_{,,}}+p_{,}\right)\cdot\overline{r_{,}q_{,,}}=r_0\cdot\left(p_{,,}q_{,}\cdot\frac{1_{,,}}{1_{,}}+p_{,}q_{,,}\right)\cdot a_{,},$$ seu

$$c_{,}\,a_{,}\,(q_{,}p_{,,}-p_{,}q_{,,})\cdot\overline{r_{,}q_{,,}}=r_0 a_{,}\cdot\overline{q_{,}a_{,,}}\left(p_{,,}q_{,}\cdot\frac{1_{,,}}{1_{,}}+p_{,}q_{,,}\right),$$ seu formæ

$\overline{r_{,}q_{,,}}=A\cdot r_0\cdot q_{,}$ et $\overline{r_{,}p_{,,}}=B\cdot r_0\cdot p_{,}$ h. e.

$r_{,}q_{,,}-r_{,,}q_{,}=Ar_0q_{,}$ et $r_{,}p_{,,}-r_{,,}p_{,}=Br_0p_{,}$ seu

$r_{,}\varkappa-r_{,,}=Ar_0$ et $r_{,}\pi-r_{,,}=Br_0$, unde $r_{,}(\varkappa-\pi)=(A-B)\,r_0$, et $r_{,,}(\varkappa-\pi)=(A\pi-B\varkappa)r_0$. Adhibitis igitur V illis rationibus, quæ vices æquationum VI reliquarum gerebant, non modo $\varkappa$, π et n, sed etiam $r_{,}:r_0$ et $r_{,,}:r_0$ invenimus; facile enim ex formis modo suppositis reperies

$$A=\frac{(\pi n+\varkappa)\,(a_{,,}-a_{,}\varkappa)}{c_{,}\,.\,(\pi-\varkappa)}\quad\text{et}\quad B=\frac{(\pi n+\varkappa)\,.\,(a_{,}\pi-a_{,,})}{n\,c_{,}\,(\pi-\varkappa)}.$$

Obtinuimus igitur $x'=\dfrac{p_{,}}{r_0}\cdot\dfrac{(x+\pi y)\,(\varkappa-\pi)}{\varkappa-\pi+(A-B)x+(A\pi-B\varkappa)\,y}$

atque $y'=\dfrac{q_{,}}{r_0}\cdot\dfrac{(x+\varkappa y)\,(\varkappa-\pi)}{\varkappa-\pi+\overline{A-B}\,x+\overline{A\pi-B\varkappa}\,y}$, seu formæ $x'=P.X$ et $y'=Q.Y$,

ubi P et Q adhuc indeterminata sunt. Sed ex $ny'dx'+x'dy'=o$, obtinetur $\dfrac{ndx'}{x'}+\dfrac{dy'}{y'}=o$ et integrando $y'x'^n=$ Cst, et igitur

$Q.YP^n.X^n=$ Cst, seu $YX^n=$ Cst, ideoque Q et P in Cst comprehenduntur. Æquationis igitur

$(a_{,}x+a_{,,}y)\,(ydx-xdy)+(b_{,}x+b_{,,}y)\,dx+(c_{,}x+c_{,,}y)\,dy=o$ integrale est

formæ $\dfrac{(x+\varkappa y)\,.\,\overline{x+\pi y}^{\,n}}{(\varkappa-\pi+\overline{A-B}x+\overline{A\pi-B\varkappa}.y)^{1+n}}=$ Cst. Si igitur breviter ponimus

$\overline{x+\varkappa y}.\overline{x+\pi y}^{\,n}.(1+\alpha x+\beta y)^m=$ Cst et $m=-1-n$, differentiando obtinemus

$(xdy-ydx).(\overline{m\beta+\overline{\varkappa+\pi n}\alpha}\,x-\overline{m\pi\varkappa\alpha+\overline{\pi+n\varkappa}\beta}\,y)\;+$

$+dy(\overline{\varkappa+\pi n}\,x-m\varkappa\pi y)+dx(\overline{\pi+n\varkappa}\,y-mx)=o$, et igitur comparando cum æquatione data, antea in r ducta,

1) $1+n=-m=rb_{,}$, $\left(\text{unde } r=\frac{1+n}{b_{,}}=\frac{-m}{b_{,}}\right)$ 2) $-m\varkappa\pi=rc_{,,}$,

3) $\varkappa+\pi n=r.c_{,}$ et 4) $\pi+n\varkappa=rb_{,,}$, unde $\varkappa=\frac{r(c_{,}-nb_{,,})}{1-n^2}=\frac{c_{,}-nb_{,,}}{b_{,}(1-n)}$, et

$\pi=\frac{r.(b_{,,}-nc_{,})}{1-n^2}=\frac{b_{,,}-nc_{,}}{b_{,}(1-n)}$. Sed jam (ex 1) et 2)) $c_{,,}=b_{,}\varkappa\pi$, ideoque erit

$b_{,}c_{,,}(1-n)^2=\overline{c_{,}-nb_{,,}}.\overline{b_{,,}-nc_{,}}$, unde dupliciter exponens n, ideoque $\varkappa$ et π, determinantur. Tandem erit 5) $m\beta+\overline{\varkappa+\pi n}\alpha=-ra_{,}$ seu

$b_{,}\beta-c_{,}\alpha=a_{,}$ et 6) $m\pi\varkappa\alpha+\overline{\pi+n\varkappa}\beta=ra_{,,}$ seu $b_{,,}\beta-c_{,,}\alpha=a_{,,}$, unde α et β facile habentur, nisi forte $c_{,}:b_{,}=c_{,,}:b_{,,}$ fuerit. *)

Hausto igitur valore alterutro ipsius n ex æquatione

$b_{,}c_{,,}.\overline{1-n}^2-\overline{c_{,}-nb_{,,}}.\overline{b_{,,}-nc_{,}}=o$, erit propositæ illius æquationis differentialis $(a_{,}x+a_{,,}y)(xdy-ydx)=(b_{,}x+b_{,,}y)dx+(c_{,}x+c_{,,}y)dy$ integrale hoc

$$\frac{(b_{,}\overline{1-n}.x+\overline{c_{,}-nb_{,,}}y).(b_{,}\overline{1-n}x+\overline{b_{,,}-nc_{,}}y)^n}{(\overline{b_{,}c_{,,}}+\overline{a_{,}b_{,,}}.x+\overline{a_{,}c_{,,}}.y)^{n+1}}=\text{Cst.}$$

Sunt vero universim valores ipsius n reciproci, seu n et $\frac{1}{n}$, qui idem in formula integrali valent. Sin vero $c_{,,}=o$ fuerit, erit n vel $=\frac{c_{,}}{b_{,,}}$ vel $=\frac{b_{,,}}{c_{,}}$, et

igitur $\frac{x^{b_{,,}}(b_{,}\overline{b_{,,}-c_{,}}x+\overline{b_{,,}^2-c_{,}^2}y)^{c_{,}}}{(c_{,}b_{,,}-\overline{a_{,}b_{,,}}x+a_{,,}c_{,}y)^{c_{,}+b_{,,}}}=\text{Cst.}$

*) Eædem vero VI æquationes aliter solutæ subministrant $b_{,}\pi\varkappa=c_{,,}$, $b_{,}(\pi+\varkappa)=b_{,,}+c_{,}$, ideoque π et $\varkappa$ radices æquationis $b_{,}\tilde{\omega}^2-\overline{b_{,,}+c_{,}}\tilde{\omega}+c_{,,}=o$, atque

$$n=\frac{b\varkappa-\pi}{\varkappa-b\pi},\quad \alpha=\frac{\overline{a_{,}b_{,,}}}{\overline{b_{,}c_{,,}}}\ \text{et}\ \beta=\frac{\overline{a_{,}c_{,,}}}{\overline{b_{,}c_{,,}}}.$$

Et si $b_{,}=\omega=o$ fuerit, erit $\dfrac{y^{c_{,}}.(\overline{b_{,,}+c_{,}}\,x+c_{,,}y)^{b_{,,}}}{(-c_{,}b_{,,}+a_{,}b_{,,}x+\overline{a_{,}c_{,,}}y)^{c_{,}+b_{,,}}}=$ Cst. Quia enim

$\overline{b_{,}c_{,,}}\,\overline{1+n^2}=(2b_{,}c_{,,}-b_{,,}^2-c_{,}^2)n$, erit evolvendo $n+\dfrac{1}{n}=h+\dfrac{1}{h}+\dfrac{\omega c_{,,}}{c_{,}^2}\cdot\left(\dfrac{1-h}{h}\right)^2$,

si $b_{,,}=h.c_{,}$ et n igitur $=h-\dfrac{\omega c_{,,}}{c_{,}^2}\cdot\dfrac{1-h}{1+h}+\ldots$; quo valore suffecto integralis forma constabit.

Sin vero $c_{,}=b_{,,}$ fuerit, erit ex æq. 3) et 4) $(\varkappa-\pi)(1-n)=o$, ideoque vel $n=1$, et $r=\dfrac{2}{b_{,}}$, $m=-2$, $\varkappa+\pi=\dfrac{2c_{,}}{b_{,}}$ et $\varkappa\pi=\dfrac{c_{,,}}{b_{,}}$, unde $\varkappa$ et π ex æquatione $b_{,}\varkappa^2-2c_{,}\varkappa+c_{,,}=o$, et $\alpha=\dfrac{a_{,,}b_{,}-a_{,}c_{,}}{c_{,}^2-c_{,,}b_{,}}$, $\beta=\dfrac{a_{,,}c_{,}-a_{,}c_{,,}}{c_{,}^2-c_{,,}b_{,}}$,

ideoque integrale hoc: $\dfrac{b_{,}x^2+2c_{,}xy+c_{,,}y^2}{(c_{,}^2-c_{,,}b_{,}+(a_{,,}b_{,}-a_{,}c_{,})x+(a_{,,}c_{,}-a_{,}c_{,,})y))^2}=$ Cst.

Vel etiam $\varkappa=\pi$, ideoque $\varkappa=\pi=\dfrac{c_{,}}{b_{,}}=\sqrt{\dfrac{c_{,,}}{b_{,}}}$, qui casus tum igitur tantummodo locum habet, cum $c_{,}^2=b_{,}c_{,,}$ (et $c_{,}=b_{,,}$) fuerit, eritque

$$\left(\frac{b_{,}x+c_{,}y}{\overline{a_{,}c_{,}-a_{,,}b_{,}}\,x+\overline{a_{,}c_{,,}-a_{,,}c_{,}}\,y}\right)^m=\text{Cst.}$$

Tandem vero si $c_{,}:b_{,}=c_{,,}:b_{,,}$ fuerit, poniturque $c_{,}=cb_{,}$, $c_{,,}=ec_{,}$ et $b_{,,}=eb_{,}$, erit æquatio data $(a_{,}x+a_{,,}y)(xdy-ydx)+(x+ey)(b_{,}dx+c_{,}dy)=o$.

Hæc vero facile juxta primum modum integratur, ponendo $y=\dfrac{1}{u}$ et $x=\dfrac{z}{u}$, quo facto erit $-b_{,}.u=(z+c)\displaystyle\int\frac{(a_{,,}+a_{,}z)\,dz}{\overline{z+e}.\overline{z+c}^2}$;

Quæ formula notissimo modo reducenda est.